

AP[®] CALCULUS BC
2012 SCORING GUIDELINES

Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W , where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F . The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

- (a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.
- (b) Use the data in the table to evaluate $\int_0^{20} W'(t) dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t) dt$ in the context of this problem.
- (c) For $0 \leq t \leq 20$, the average temperature of the water in the tub is $\frac{1}{20} \int_0^{20} W(t) dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20} \int_0^{20} W(t) dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.
- (d) For $20 \leq t \leq 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t} \cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a)
$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6}$$

$$= 1.017 \text{ (or } 1.016)$$

The water temperature is increasing at a rate of approximately 1.017°F per minute at time $t = 12$ minutes.

(b)
$$\int_0^{20} W'(t) dt = W(20) - W(0) = 71.0 - 55.0 = 16$$

The water has warmed by 16°F over the interval from $t = 0$ to $t = 20$ minutes.

(c)
$$\frac{1}{20} \int_0^{20} W(t) dt \approx \frac{1}{20} (4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$$

$$= \frac{1}{20} (4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$$

$$= \frac{1}{20} \cdot 1215.8 = 60.79$$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

(d)
$$W(25) = 71.0 + \int_{20}^{25} W'(t) dt$$

$$= 71.0 + 2.043155 = 73.043$$

2 : $\begin{cases} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{cases}$

2 : $\begin{cases} 1 : \text{value} \\ 1 : \text{interpretation with units} \end{cases}$

3 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \\ 1 : \text{underestimate with reason} \end{cases}$

2 : $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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Question 2

For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

- (a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.
- (b) Find the x -coordinate of the particle's position at time $t = 4$.
- (c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.
- (d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

(a) $\left. \frac{dx}{dt} \right|_{t=2} = \frac{2}{e^2}$

Because $\left. \frac{dx}{dt} \right|_{t=2} > 0$, the particle is moving to the right at time $t = 2$.

$$\left. \frac{dy}{dx} \right|_{t=2} = \frac{\left. dy/dt \right|_{t=2}}{\left. dx/dt \right|_{t=2}} = 3.055 \text{ (or 3.054)}$$

(b) $x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t} dt = 1.253 \text{ (or 1.252)}$

(c) Speed = $\sqrt{(x'(4))^2 + (y'(4))^2} = 0.575 \text{ (or 0.574)}$

$$\begin{aligned} \text{Acceleration} &= \langle x''(4), y''(4) \rangle \\ &= \langle -0.041, 0.989 \rangle \end{aligned}$$

(d) Distance = $\int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2} dt$
 $= 0.651 \text{ (or 0.650)}$

$$3 : \begin{cases} 1 : \text{moving to the right with reason} \\ 1 : \text{considers } \frac{dy/dt}{dx/dt} \\ 1 : \text{slope at } t = 2 \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

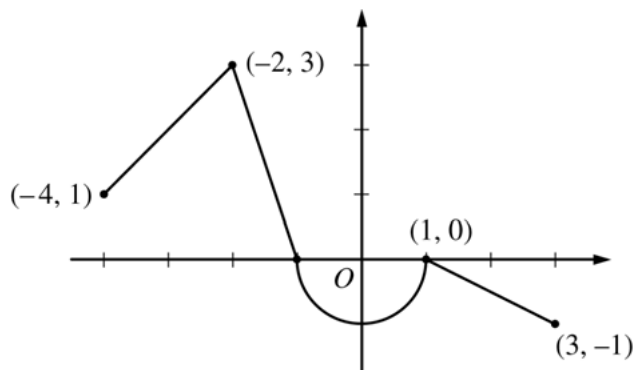
$$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration} \end{cases}$$

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

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Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t) dt$.



Graph of f

- (a) Find the values of $g(2)$ and $g(-2)$.
- (b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.
- (c) Find the x -coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.
- (d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t) dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$
 $g(-2) = \int_1^{-2} f(t) dt = -\int_{-2}^1 f(t) dt$
 $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

2: $\begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$
 $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

2: $\begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

3: $\begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

2: $\begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f' , the derivative of f , are given for selected values of x in the table above.

- (a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.
- (b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x) dx$. Use the approximation for $\int_1^{1.4} f'(x) dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.
- (c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.
- (d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15, f'(1) = 8$

An equation for the tangent line is
 $y = 15 + 8(x - 1)$.

$$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$$

(b) $\int_1^{1.4} f'(x) dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$$f(1.4) = f(1) + \int_1^{1.4} f'(x) dx$$

$$f(1.4) \approx 15 + 4.6 = 19.6$$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$$

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$
 $= 15 + 8(x - 1) + 10(x - 1)^2$

$$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$$

2 : $\begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

3 : $\begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

2 : $\begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$

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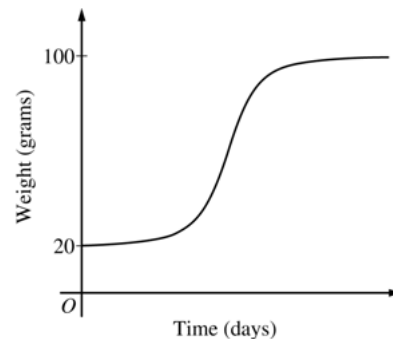
Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

- (a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.
- (b) Find $\frac{d^2B}{dt^2}$ in terms of B . Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.
- (c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left. \frac{dB}{dt} \right|_{B=40} = \frac{1}{5}(60) = 12$

$$\left. \frac{dB}{dt} \right|_{B=70} = \frac{1}{5}(30) = 6$$

Because $\left. \frac{dB}{dt} \right|_{B=40} > \left. \frac{dB}{dt} \right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5} \frac{dB}{dt} = -\frac{1}{5} \cdot \frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \leq B < 100$. A portion of the given graph is concave up.

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$$\int \frac{1}{100 - B} dB = \int \frac{1}{5} dt$$

$$-\ln|100 - B| = \frac{1}{5}t + C$$

Because $20 \leq B < 100$, $|100 - B| = 100 - B$.

$$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$$

$$100 - B = 80e^{-t/5}$$

$$B(t) = 100 - 80e^{-t/5}, \quad t \geq 0$$

2 : $\left\{ \begin{array}{l} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{array} \right.$

5 : $\left\{ \begin{array}{l} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{array} \right.$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

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Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \dots$$

- (a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g .
- (b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.
- (c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left| \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} \right| = \left(\frac{2n+3}{2n+5} \right) \cdot x^2$

$$\lim_{n \rightarrow \infty} \left(\frac{2n+3}{2n+5} \right) \cdot x^2 = x^2$$

$$x^2 < 1 \Rightarrow -1 < x < 1$$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \leq x \leq 1$.

(b) $\left| g\left(\frac{1}{2}\right) - \frac{17}{120} \right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \dots + (-1)^n \left(\frac{2n+1}{2n+3} \right) x^{2n} + \dots$

5 : $\left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies interior of} \\ \quad \text{interval of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \text{uses the third term as an error bound} \\ 1 : \text{error bound} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \text{first three terms} \\ 1 : \text{general term} \end{array} \right.$